Optimal Betting

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You have just calculated that the San Diego Chargers are 60% sure to beat the Oakland Raiders in an upcoming playoff game. The sports book shows the game even, so you don't have to give up any points if you bet on either team. You must give odds, however, since the sports book pays only 10 to 11 on winning bets. You have a $10,000 betting bankroll. Now you ask yourself, "How much should I bet on this game?" If your answer is about $1600, you may want to skip the rest of this article. Otherwise, read on.

Betting on the outcome of a game is a game in itself. We call such a game favorable if we are more likely to win money than to lose money, so a bet on the Chargers appears to be a favorable game. We'll have to check on that, however, since odds are involved. Those who bet on the Raiders are in an unfavorable game, if the 60% Charger win probability is correct. The sports book handicapper believes the Charger-Raider match is a "fair" game, in which neither side has an advantage. If the teams are really evenly matched, the 10 to 11 payoff makes the betting game unfavorable to bettors on both sides. That is how a sports book makes money. If you were to make such a bet with a friend at even money odds, however, you would be in a favorable betting game.

Game probabilities are usually expressed in terms of p, q, and r. The letter p stands for the probability of success, with p = 0 if there is no chance of winning and p = 1.0 when a win is certain. If the Chargers are 60% sure to beat the Raiders, then p = .60. The letter q represents the probability of failure, which also has a range of 0 to 1.0. If there is no chance of a tie, true in this case, then q = .40 for the Charger bet. The letter r designates the probability of a tie. If the Charger-Raider game were a regular season contest, a tie would be possible. If there is a 2% chance of a tie, then r = .02. Since p + q + r must total 1.0, the probabilities for winning, losing, and tying with a bet on the Chargers would then be p = .60, q = .38, and r = .02, if we say that the Chargers' chance of winning is still 60%.

To complete the description of the betting game we need to state the payoff for a win compared to the loss "payout," or in other words, the odds offered for the wager. For these we use the Greek letters α and β, α standing for the win payoff and β for the loss payout (the total wager). The ratio of α to β tells what the betting odds are. For a football wager in a sports book, the odds offered are generally 10 to 11, so α = 10 and β = 11. To win 10, you must risk 11. We now have values for all the parameters governing a bet on the Chargers in the playoff game: p = .60, q = .40, r = 0, α = 10, and β = 11.

Expected Gain

How much is this bet worth? We'll use the term "expected gain" for this purpose. Expected gain is not the same as "edge," however. That comes next. Expected gain (E) is the payoff (α) times the win probability (p), minus the payout (β) times the loss probability (q):

$$E = \alpha p - \beta q$$

For our football game, the calculation is E = (10)(.60) - (11)(.40) = 1.6. You are risking 11 to win 10, with a 60% chance of success and a 40% chance of failure. Eleven what? Ten what? It doesn't matter. Whether they are one-dollar bills or hundred-dollar bills, you may expect to win 1.6 of them. Let's call them units. "Expect" means that if you were to bet a huge number of games identical to this one, the average of all your wins and losses would be very close to 1.6 units per game. You can't win 1.6 units on one game, of course, just as you can't have an average number of children (1.7). You either win 10 or lose 11, but the expected gain is 1.6.

When solving the above equation, if E turns out to be zero or a negative value, you don't place a bet unless you're a "gottabet" who must bet every game. Suppose the Chargers had only a slightly better than even chance of winning, say 52%. Then p = .52, q = .48, and E = (10)(.52) - (11)(.48) = -.08. The expected gain turns out to be an expected loss, so you skip this game. You need to have p a little larger to consider a bet at the odds offered, and a lot larger to make a bet really worthwhile. For a 10 to 11 payoff, the expected gain is close to zero when p = .524 and there is no tie possibility.

The Edge

Now, is 1.6 the "edge" (advantage) for the Charger bet as originally stated? Obviously not, although many writers on gambling have the habit of treating edge as synonymous with expected gain. No, the edge, or advantage (A), is the ratio of expected gain (E) over amount risked (β):

$$A = E/\beta = (\alpha p - \beta q) / \beta$$

We have calculated that $\alpha p - \beta q$, the expected gain, is 1.6 for the Charger bet, so the advantage is 1.6/11, or .1455. In percentage terms this is a trifle more than a 14.5% edge. For every dollar you risk on this football game, you may expect to win 14.5 cents. Here too, "expect" means an anticipated average result if the same bet could be made on identical games a huge number of times.

For even-money wagers, as when betting with a friend, the win and loss amounts are equal ($\alpha = \beta$) and the advantage equals the expected gain for a one-unit bet. Both would have a value of p - q, which for the Charger bet is .60 - .40 = .20. Thus, the friendly wager on the Chargers would have a 20% edge instead of 14.5%, the difference representing the sports book's share of the action.

Bankroll Growth Rate

All right, so you're betting 11 units to win 10, but you want to know how large the units should be in terms of dollars. The question is not immediately answerable, because the appropriate amount depends on your personal financial goals at the moment. Do you need a lot of money fast? Then perhaps you should bet the entire $10,000 bankroll. That would put you out of business with a loss, however. Maybe you would be content to have a 95% chance of not going broke in the course of making 20 bets identical to this one. Then you must bet a smaller amount, about $1350.

Instead of one of these rather subjective aims, let's assume that your goal is more objective: To maximize your bankroll's probable rate of increase over the long haul, with a minuscule chance of ever going broke.

Bankroll growth rate, or rate of return, is analogous to a compound interest rate. If you have a bank account that carries a 6% interest rate, compounded annually, then the account will grow by a factor of 1.06 each year. After two years the total will equal the original deposit times 1.06 times another 1.06, which comes to 1.1236. To show the growth factor after many years we use an exponent instead of writing 1.06 umpty-ump times. For instance, after ten years the account would grow by a factor of 1.06^{10}, about 1.79 times the original amount. The .06 value is the annual rate of return, and 1.79 is the growth factor after ten years.

It is intuitively obvious, and true, that the only way to get a healthy bankroll growth rate with safety is to always bet a fraction of current bankroll that is related to the "favorability" of the wager. We call this policy "fractional betting." No other betting strategy, no progressive betting scheme, no flat bet philosophy, can match the expected bankroll growth rate achievable with fractional betting.

Optimal Bet Size

But what fractions of current bankroll are proper for advantageous bets? Assuming that you will have average luck (the number of wins and losses will equal expected values), then you probably want fractions that will result in a maximum rate of return for average luck.

Rate of return is normally thought of as a rate per unit of time (per year, in the bank account example). In a gaming context, however, it's a rate of return per play of a game. If the win/loss ratio is the expected p/q, the equation for the expected rate of return (R) is:

$$R = (1 + \alpha f)^p (1 - \beta f)^q - 1,$$

where f is the fraction of bankroll represented by one betting unit.

For the Charger wager you are betting 11 such units, with $p = .6$, $q = .4$, $\alpha = 10$, $\beta = 11$. You want to choose a fraction f that will result in a maximum value for R. This optimal fraction will be designated as f^*, which can be derived from the last equation by any first-year calculus student:

$$f^* = p/\beta - q/\alpha$$

This is equivalent to $f^* = (\alpha p - \beta q) / \alpha \beta = E/\alpha \beta$, the expected gain divided by the product of the odds. Since the advantage A is equal to $(\alpha p - \beta q) / \beta$, we can also say that the optimal fraction is equal to the advantage divided by the payoff: $f^* = A/\alpha$. You will sometimes read that f^* is equal to the edge, but that is true only when the odds are even ($\alpha = \beta$) and there are no tie possibilities. In such a game, the values of f^*, E, and A are all equal to p - q.

The edge for the Charger bet is .145 and the payoff is 10, so the optimal fraction (f^*) of bankroll for a betting unit in this game is .145/10 = .0145. You are betting 11 times this fraction ($\beta = 11$), so the

optimal wager is (.0145)(11)($10,000), about $1600. If you were to bet on this game with a friend at even odds, the value of f^* would be $p - q$, which for $p = .6$ and $q = .4$ is .2, making the optimal bet $2000. This is both the unit size and the bet size, since $\alpha = \beta = 1$.

Going back to the sports book bet, what rate of return for average luck can you expect from the $1600 bet? To find out, just plug 0.0145 (f^*) into the equation for expected rate of return:

$$R = [1 + (10)(.0145)]^{0.6} [1 - (11)(.0145)]^{0.4} - 1 = .0118$$

If you were to make this identical wager many times and figure out the rate of return that results after 60% wins and 40% losses, the value for R would be .0118. That makes the growth factor 1.0118^n after n such wagers. Note that the order of wins and losses makes no difference. Their ratio, not their order, determines the final result. This may or may not be a comforting thought during a losing streak. Contrast this with progressive betting schemes (e.g., double the bet after every loss, cut back on a win), in which the order of wins and losses may be very important.

Are you wondering about the validity of the equation for rate of return? Try it with whole numbers instead of p and q for exponents. In ten games with $p = .6$ and $q = .4$, you figure to win six times and lose four times, right? If so, the bankroll growth factor after ten games is $[1 + (10)(.0145)]^6 [1 - (11)(.0145)]^4$, which comes to 1.1245, bringing your bankroll of $10,000 up to $11,245. Doesn't seem like much, does it? However, try any other fraction of bankroll for the betting unit, replacing .0145. You will not find one that produces a greater rate of return for expected luck. That is why we call f^* the optimal fraction, and the strategy of betting f^* units "optimal betting." This betting policy is optimal under two criteria according to Leo Breiman (whose book on the subject I cannot find, so I got this second hand): (1) minimal expected time to achieve a fixed level of resources, and (2) maximal rate of increase of wealth.

To verify that the 1.1245 growth factor after ten games represents a rate of return (R) of .0118, we use logarithms:

$$\log (1 + R) = (\log 1.1245)/10 = .011734$$

$1 + R$ is therefore the antilog of .011734, which is 1.0118, making R the number we were looking for: .0118.

Can we express the expected rate of return for optimal betting (R_{max}) in terms of p and q? Certainly. We know already that f^* bets figure to produce R_{max}:

$$R_{max} = (1 + \alpha f^*)^p (1 - \beta f^*)^q - 1$$

Substituting $(\alpha p - \beta q)/\alpha\beta$ for f^*, and using the fact that $p + q$ is 1.0 in a game with no ties, a little algebra produces an equation for R_{max} in terms of p, q, α, and β:

$$R_{max} = [p(\alpha + 1)]^p [q(\beta + 1)]^q / \alpha\beta$$

For even-money unit bets ($\alpha = \beta = 1$) this simplifies to:

$$R_{max} = (2p)^p (2q)^q - 1$$

Applying this equation to the "friendly" charger bet, we get $R_{max} = [(2)(.6)]^{0.6} [(2)(.4)]^{0.4} - 1 = .02$. The resultant expected bankroll growth rate for this bet, $1 + R_{max}$, is therefore 1.02. Mathematicians prefer to talk about maximizing the logarithm of $1 + R_{max}$, which they call G. I guess this is because optimal betting is equivalent to having a logarithmic utility function (doubling one's bankroll is twice as likely as losing half of it). Anyway, G_{max} is easily obtained from the previous equation:

$$G_{max} = \log(1 + R_{max}) = p\log 2p + q\log 2q$$

Games with Ties

Now, how do ties (r not equal to zero) affect our calculations? For one thing the previous f* equation won't work. We must first change the p and q values, dividing each by their sum:

$$p' = p / (p + q) \quad q' = q / (p + q)$$

For the Charger bet with 2% chance of a tie, $p' = .60/.98 = .612$, and $q' = .38/.98 = .388$. The optimal bankroll fraction (f*) for a betting unit when ties are possible is:

$$f* = (\alpha p' - \beta q') / \alpha\beta$$

For the Charger sports book bet, $f* = [(10)(.612) - (11)(.388)] / 10 / 11 = .017$. For the friendly wager, for which $\alpha = \beta$, $f* = p' - q' = .612 - .388 = .224$. Here are the resultant respective rates of return:

Sports book: $R_{max} = (1.017)^{0.6} (.983)^{0.38} - 1 = .0036$

Friend: $R_{max} = (1.224)^{0.6} (.776)^{0.38} - 1 = .0252$

It is obviously much better to bet with a (non-welshing) friend than with a sports book! Note that we use the original p and q values, not p' and q', for exponents in the R_{max} equation.

In terms of p and q, the equation for R_{max} in an even-money payoff game with tie chances becomes:

$$R_{max} = (2p')^p (2q')^q - 1$$

And for the mathematicians:

$$G_{max} = \log (1 + R_{max}) = p\log 2p' + q\log 2q'$$

Optimal vs Flat Betting

"Wait," you may say, "How about flat betting $1600 every time instead of betting a fraction of current bankroll? The edge is .1455, so after ten flat bets of $1600 I expect to win $1600 times .1455 times 10, which is $2328. That gives me a total expected bankroll of $12,328 for ten games instead of the $11,245 you got by so-called optimal betting. Why not flat bet instead?"

Yes, but ten games is not the "long haul." Let's look at 100 games. Now flat betting $1600 wins an expected $23,280, resulting in a bankroll of $33,280. This assumes you can play on credit, because there is a 15% chance you will be down $10,000 or more during the 100 games. The optimal bettor's bankroll with expected luck is $10,000 times 1.0118^{100}, or $32,320. Almost the same result as for flat betting, but with no chance of going broke (since only a fraction of bankroll is used for every bet).

Now go on for a second 100 games. Flat betting (with credit) gives you an expected win of $46,560, for a total bankroll of $56,560. Assuming the same luck, optimal betting yields $10,000 times 1.0118^{200}, which is $104,456. Much better!

The principle here is analogous to compounded vs simple interest. A compounded rate is eventually going to get you more money, even if a simple interest rate is greater. Assume a game with a small edge, with repeated plays by both a flat bettor and an optimal bettor. The flat bettor makes the same bet as the optimal bettor on the first play (and figures to win the same), but thereafter he continues to make identical bets. The optimal bettor wagers a constant fraction (f*) of his current bankroll. The optimal bettor does worse for a while in a typical game, but after a sufficient number of plays (depending on p, q, and r values) he passes the flat bettor. From then on his bankroll grows dramatically while that of the flat bettor lags further and further behind

Non-Optimal Bet Fractions

What happens to a fractional bettor who bets more or less than the f* fraction? With smaller bet fractions the probable growth factor for bankroll increases as the fraction increases, topping out when the fraction reaches f*. For fractions greater than f*, the probable growth factor starts decreasing with increasing bet fractions, becoming 1.0 (no growth) at or near 2f* bets. For fractions larger than 2f*, the probable growth factor is less than 1.0 (bankroll shrinking), approaching zero as a limit. When a fraction of bankroll is bet every time, it is theoretically impossible to actually reach zero (but the bankroll may become less than the size of a minimum permissible bet!).

To summarize: If you have the expected amount of luck (actual win/loss ratio is p/q), your bankroll grows fastest with f* fractional bets, stays about the same with bets of 2f*, and actually shrinks for a policy of betting more than 2f*. Another item of interest is that equal degrees of overbetting and underbetting on each side of f* results in equal (reduced) bankroll growth. You can't make up for overbetting by some compensatory underbetting. They don't compensate, so far as probable growth factor is concerned.

Also interesting are the bankroll fluctuations that occur over time for variations in bet fractions. Although bets of f*/2 and 1.5f* result in the same expected bankroll growth rate (less than that for f*), the smaller fraction will see a much smaller fluctuation in bankroll along the way. Betting 1.5f* brings exhilarating/scary swings in bankroll size over time. Mixing f*/2 bets and 1.5f* bets would

produce fluctuations somewhere in between, so they do compensate somewhat in that regard (but not symmetrically).

With 2f* fractional bets the results for a large number of players will be wildly different, some making a fortune, others losing almost the entire bankroll. The median player in the ranking of results should have close to the same bankroll he started with, although his bankroll has probably fluctuated greatly along the way. Above 2f* the median player figures to lose money and the fluctuations become even greater.

The Median Fallacy

The above analysis seems to say that a fractional bettor will lose money in a coin-tossing game if he wins the expected 50% of the time. Since the edge is zero in such a fair game, f* is also zero. Any bet fraction exceeds 2f* and will lead to eventual shrinkage of bankroll if wins and losses balance. Look at the rate of return for expected luck when betting 10% of bankroll on every coin toss:

$$R = (1.1)^{0.5} (.9)^{0.5} - 1 = -.005$$

Even in a favorable game, the rate of return for average luck will be negative if the bet fraction is greater than 2f*. Suppose you risk 50% of your bankroll in the "friendly" Charger bet, for which f* is .20, or 20%:

$$R = (1.5)^{0.6} (0.5)^{0.4} - 1 = -.0334$$

Amazing! The fractional better appears to lose money in favorable games if he continually bets more than the 2f* fraction. But hold on a minute! How could any betting scheme in a fair or favorable game lead to an expected loss? A basic theorem of gambling says that all betting methods have the same mathematical expectation in the long run. Come to think of it, something else looks odd. For the "friendly" Charger bet, f* is 20% of bankroll and the edge is 20%, so we expect to win 0.2 times 0.2 times the current bankroll = .04 times current bankroll when we make this bet. But the R_{max} we calculated for this game is only .02, half of .04. Do we have a fallacy here?

Yes and no. Saying that a fractional bettor's win rate (total winnings vs total wagers) is only half a flat bettor's win rate is true in a way, but only for the median result, not for the average result. It's the median fallacy.

Suppose you have two teams of 1001 players each, all with identical starting bankrolls, and all betting on the same favorable propositions over a period of time. Members of one team continually flat bet an amount equal to the original optimal bet. The other team bets the f* fraction of current bankroll on every proposition. The median result for each team--that of the 501st person in ranking of results--will indeed show the median fractional bettor's win rate to be about one-half that of the median flat bettor's rate. In fact, the comparison gradually worsens for the fractional bettor as the number of plays increases. But if you add up all the 1001 results for each team, you will find that each team has made about the same win rate on total wagers. That is, total money won divided by total money bet will be the same for both teams. As stated before, no betting scheme can change that number.

You may have noticed my continual references to "return for average luck" or (the same thing) "return for expected luck." In the case of fractional betting, this is not the same as "return for expected win." Expectation may be defined as the average result for an arbitrarily large number of players. Some will have good luck, some bad, and some (very few) average luck. For flat bettors, who bet the same number of dollars each time, average luck yields an average win total. For fractional bettors, however, the result for average luck is not the average result!

Take a coin-tossing game. Suppose a fractional bettor and a flat bettor start out with the same bet, say 10% of bankroll. Thereafter the fractional better always bets 10% of current bankroll, while the flat bettor makes identical bets for each toss. With average luck, the fractional bettor figures to lose money and the flat bettor breaks even. With good luck, the fractional bettor's bankroll rises exponentially, while a flat bettor's bankroll-to-luck relationship continues linear with luck. Equal degrees of luck on both side of average are equally probable, so for fractional betting the average result is greater than the result for average luck. With very bad luck, the fractional bettor loses less than the flat bettor, who may well go broke along the way. With very good luck, the fractional bettor may win a bundle. With average luck, the flat bettor breaks even while the fractional bettor loses.

If you take *all* possible degrees of luck, compute the fractional bettor's win for each, multiply each by its probability, and add it all up, what do you get? Zero, of course, the same as for flat betting. It's a coin-tossing game, isn't it? Zero has to be the average (expected) result for any fair game, even though for fractional bettors it is not the result for average (expected) luck.

But what about the "long haul?" Isn't everyone's luck going to be the same after a jillion plays? In a sense, yes. Everyone's luck is going to average out close to expected luck, *percentagewise*, as the number of plays gets huge. In absolute terms, however, the probable difference between actual number of wins and the expected number increases with number of pays. At the same time, the probability of having exactly average luck decreases.

In a coin-tossing game, one standard deviation for 100 tosses is +/-5%. This means that 68% of players in such a game will win between 45 and 55 times out of 100, within five wins of the 50-win average. For 10,000 tosses, however, the standard deviation is much smaller, +/- .05%. But 0.5% of 10,000 is 50, so the spread of one standard deviation is 4,950 to 5,050 wins, ten times greater than for 100 tosses. Very few fractional bettors will have the average 5,000 wins, resulting in a negative rate of return. Of the rest, half will do better, half worse. Some of those who do better will do sensationally well. Those who do worse will never go broke, theoretically, because they are always betting a fraction of current bankroll. The overall average result, however, remains at the immutable expectation for coin-tossing: exactly zero.

Of course all this is only of academic interest, since there is no incentive for betting in a game that isn't favorable.

A Computer Simulation

Is it best to assume we're going to have average luck and bet on that basis? Not necessarily. If assumption of average luck were the optimum course in life, we would never buy auto insurance or spare tires. Besides, while average luck is the most probable outcome of many wagers, it is very improbable when compared to the probability of having better or worse luck.

To pursue this further, let's examine the following game: p = .51, q = .49, r = 0, 1-1 odds ("even money."). The edge is 2% and f* is the same, .02. I fed this game into a computer and recorded the results of 100 simulated players, each starting with a $1000 bankroll and each playing the game 5000 times with f* bets (2% of current bankroll). A random number generator determined the outcome of each play. I confess that I expected the average of all bankrolls, after 5000 bets, to have grown according to the equation for R_{max}:

$$R_{max} = [(1.02)^{.51} (.98)^{.49} - 1 = .0002$$

The average ending bankroll for all players, I thought, would show a bankroll growth of 1.0002^{5000}, which is 2.718 times the starting bankroll, or $2718. To my surprise, the average came to almost $7000. Had I done something wrong? Then it came to me: Of course! The average result is not the result for average luck! It is true for flat betting, but not for fractional betting. Of the 100 simulated players, those with close to expected luck (2550 wins) did indeed have final bankrolls of around $2700. About half the remaining players did worse, half better. The worst result was a final bankroll of $44.15, the best a whopping $78,297. You see what happens? The lucky ones pull the average up much more than the unlucky ones pull it down.

For 100 players flat betting $20 (the fractional bettor's original bet), we don't need a computer. They will win close to 100 x $20 x .02 x 5000 = $200,000. That's an average of $2000 per player, which added to the original bankroll of $1000 comes to $3000. Compare that to the $7000 average of the optimal betting players. Quite a difference!

Variations of the computerized game were very enlightening. I next tried 2000 players instead of 100. The average final bankroll was $7122. The reason for the increase is that there is a very small percentage of players who do extremely well. The larger number of players makes for a better representation of the lucky ones. The luckiest of the 2000 ended up with $161,568, while the player with my sort of luck had only $24 left. As in the case of 100 players, approximately one-fourth lost money.

That's a sobering thought, isn't it? With optimal betting, there was only a 75% chance of being ahead after 5000 plays of a game with a 2% edge. This is roughly equivalent to 5000 single-deck Blackjack bets with a "true count" of plus four. No wonder some Blackjack players get discouraged!

My next step was to vary bet fraction, going from .01 to .05 times current bankroll (f*/2 to 2.5f*). I also varied the number of plays: 1000, 3000, and 5000. The results are in Table I. Note how the average final bankroll continues to rise with bet fraction. Amazingly (to me, anyway), this is true all the way up to betting 100% of bankroll with every play! The few who never lose win such a googol of money that they bring the average up high enough to overcome the fact that everyone else goes broke.

STARTING BANKROLL $1000	Number of Plays		
CONSTANT BET FRACTION	1000	3000	5000
.01 (f*/2)	$1206 (648)	$1830 (436)	$2710 (313)
.02 (f*)	$1474 (778)	$3163 (625)	$7122 (494)
.03 (1.5f*)	$1799 (867)	$5378 (770)	$18331 (725)
.04 (2f*)	$2171 (1045)	$11549 (998)	$43832 (1004)
.05 (2.5f*)	$2557 (1147)	$16424 (1217)	$104818 (1259)

Table I. Average Final Bankroll, 2000 Players, 2% Edge
(Number of Losers in Parentheses)

Now look at how the number of losers (in parentheses) changes with bet fraction and number of plays. Small bet fractions increase the probability of being ahead after a given number of plays. For fractions less than 2f*, the number of losers appears to approach zero as the number of plays goes to infinity. The decrease toward zero is faster for small fractions, slower for larger fractions.

At about 2f* the number of winners and losers stays about the same for any number of plays. No matter how long you play with 2f* bets, you have close to just a 50% chance of being ahead (or behind) at any time.

Above 2f*, the probability of being a loser increases in both directions. Higher fractions or more plays both bring more losers, until you reach the ultimate of an infinite number of players betting 100% of bankroll forever. At this point we have the strange result I mentioned before: Although the chance of being a winner is minimal after n plays (p^n), the expected bankroll growth factor is maximal $(2p)^n$.

How did I get $(2p)^n$? To calculate the growth of the average final bankroll for a jillion players, you take the advantage (A) times the fraction (f) of a unit, times number of units risked (β) to get average bankroll growth per play. The growth rate is therefore $1 + Af\beta$ for each play, and the growth factor after n plays is $(1 + Af\beta)^n$. For 5000 plays of the 2% edge game the expected growth factor for bankroll is $1 + [(.02)(.02)(1)]^{5000}$, which is 7.386, bringing the bankroll to $7386. My computer program result of $7122 is slightly less because even 2000 players are not enough to get an accurate average with Monte Carlo programming.

Anyway, if the wager size is the entire current bankroll, the expected bankroll growth factor after n consecutive winning plays is $(1 + A)^n$, in this case 1.02^{5000}, or $(2p)^n$. Put another way, there is a $.51^n$

chance of not busting after n consecutive plays. That is p^n. If you haven't busted, your bankroll growth factor is 2^n. The expected bankroll growth factor is therefore p^n times 2^n, or $(2p)^n$.

Getting back to Table I, its lessons look rather paradoxical:

-- Betting less than f* units reduces expected winnings, but increases the probability of being a winner.

-- Betting greater than f* units increases expected winnings, but also increases the probability of being a loser.

-- With 2f* units, the chances of being a winner at any time are about 50-50, but expected winnings are greater yet.

-- With bets greater than 2f*, the probability of being a loser approaches certainty as the number of plays increases, although expected winnings become astronomical in size.

What's Best for You?

How can we apply these conclusions in a practical way? The answer is a subjective, personal decision. If you want to increase your chances of being a winner at any time, you bet less than f* units, perhaps f*/2. You will be more sure of making money than with f* units, but the winnings may not be great. If you want to increase those probable winnings, you bet more than f* units, perhaps 1.5f*. You will be less sure of winning, but you could win a lot. Finally, there is the middle road of f* unit bets, with an attractive combination of best probable bankroll growth rate and a good chance of being a winner.

How much of a chance? With f* unit bets, there is a 67% probability of doubling your money before losing half of it. This compares with 89% for f*/2 bets and 56% for 1.5 f*bets. You must pick what feels right for you. To what degree are you willing to sacrifice probable income in order to gain safety, or *vice versa*? To some degree the answer will probably depend on your bankroll size. A large bankroll may provide quite a comfortable income with a mere f*/2 unit bets, while a small bankroll owner might be tempted to wager with 1.5f* bets until his bankroll grows considerably. Maybe he can't live on the safer income of f* units. If it doesn't work out, he will just have to get a job. It follows that your betting practices may vary with bankroll size, according to your wants and needs.

Optimal vs Optimum

What is this word "optimal"? Why not "optimum"? Because I'm trying to make a distinction. An *optimal* bet fraction (f*) for a betting unit will bring the greatest rate of return if the ratio of wins and losses is the expected p/q. An *optimum* fraction is the one that brings the greatest rate of return (R_A) for the actual win/loss (W/L) ratio, which may not be p/q:

$$R_A = (1 + f)^W(1 - f)^L - 1$$

For the 2% edge game, you expect 510 wins and 490 losses in 1000 plays, and the *optimal* fraction (f*) is .02. If you actually experience 505 wins and 495 losses, the *optimum* fraction would have been .01, which is a conservative f*/2, with a rate of return of:

$$R_A = (1.01)^{505}(0.99)^{495} - 1 = .051$$

An optimal bet of .02 times bankroll would have produced a zero return:

$$R_A = (1.02)^{505}(0.98)^{495} - 1 = 0$$

In the other direction, the optimum fraction for better than average luck will be greater than f*. For 515 wins and 485 losses, the optimum bet is 0.3 times bankroll (1.5f*):

$$R_A = (1.03)^{505}(0.97)^{495} - 1 = .568$$

Compare this to an optimal f* bet:

$$R_A = (1.02)^{515}(0.98)^{485} - 1 = .492$$

Since we can't know what the optimum is going to be, we choose the most likely ratio for W/L, which is p/q. A bet fraction based on p and q (f*) will be optimal, but it probably won't be optimum.

Alternative Wagers

Suppose you are offered a choice among various favorable wagers. You are allowed to make one bet only. Is it easy to choose? Yes, if all potential wagers are offered at the same odds and the same-size bet. You naturally pick the one that has the best chance of winning. But what if the odds offered are different, the win/loss probabilities are different, and/or there is a specified bet size, perhaps different for each? Let's take an example:

A friend offers you a choice of two wagers: Game 1 for $1000 at even money, Game 2 for $1500 and you get odds of 3 to 2. You calculate that Game 1 is 10% in your favor: $p = .55$, $q = .45$, with an expected gain of $100. Game 2 is 14% against winning, $p = .43$, $q = .57$, but you are getting 3 to 2 odds, so your expected gain is $112.50. Dividing this by the amount risked, $1500, gives an edge of 7.55%. Game 1 has the better edge, but Game 2 has a greater expected gain. Which bet should you take with your $10,000 bankroll? To find out, let's look at the rate of return for each.

For Game 1 f* is $.55 - .45 = 0.1$, so the $1000 is an f* bet. The expected rate of return (R_{max}) is $(1.1)^{.55}(0.9)^{.45} = .005$. For Game 2 f* must be calculated using the odds:

$$f* = [(3)(.43) - (2)(.57)] / (3)(2) = .025$$

Betting two f* units to win three f* units would bring an expected rate of return of:

$$R_{max} = [(1 + 3(.025)]^{.43} [1 - 2(.025)]^{.57} - 1 = .0119$$

With the stipulated bet of $1500, however, f is 1500/10000/2 = .075 (3f*!), and the expected rate of return brings bankroll shrinkage:

$$R = [(1 + 3(.075)]^{.43} [1 - 2(.075)]^{.57} - 1 = -.0054$$

You bet on Game 1, because not only is the expected rate of return better, but the fraction of bankroll for a betting unit is f*, making this bet much safer. Those who choose a bet solely on the basis of expected gain may prefer Game 2. That would be a mistake in this case, if one wants to be an optimal bettor.

A richer person might rightly choose Game 2. For a $30,000 bankroll, Game 2 represents an f* bet (.025 x 30,000 x 2 = 1500). The rate of return, already calculated, is .0119. The $1000 Game 1 bet would be only 1/30 of this bankroll, giving a rate of return of $R = (1 + .0333)^{.55} (1 - .0333)^{.45} - 1 = .0028$. This 1/3 f* bet would be overly conservative for most gamblers, who would choose the alternative wager (an f* bet).

When alternative wages have equal rates of return, you choose the one for which the required bet represents a fraction of bankroll vs f* that is closer to your betting philosophy. Suppose two wagers have an expected rate of return of .08, but one requires a bankroll fraction bet of 0.8f* units, the other 1.2f* units. You then have a choice between a conservative bet and an aggressive bet. It's your decision, no one can make it for you.

When the expected rates of return for alternative wagers are not equal, you usually pick the one that has the greater rate of return, but not necessarily. If the risk factor (in terms of f/f*) for the better rate of return is unacceptably high, you may go for the safer bet. Again, it's a subjective decision. If f/f* is too high for both wagers, you might decide not to bet at all.

Forced Wagers

There are times when we are forced to select between two wagers, both of which are unfavorable. Forced? Yes. If you own a house, you *must* choose between two bets. One is with the insurance company. You can annually bet a certain β amount (the premium) that your house will burn down. The insurance company will take the bet, offering huge odds on the proposition that the house will not burn down. We can expect a negative expectation, since insurance companies make money.

Alternatively, you can bet with Lady Luck, taking the "not burn" side of the proposition by not insuring. Since α = 0 (there is no payoff for a win), this also has a negative expectation. If you can calculate the chances of your house burning down, just plug that number into the equation for rate of return, along with all the other numbers, and choose the wager that brings the better (less negative) rate of return. In this case, "bankroll" is your net worth. If your house value is at all commensurate with that net worth, insurance is probably the better bet.

Compound Games

In a compound game you must make some minimum bet on each game in a series, even though the expectation is frequently negative, zero, or so small that f* represents a bet that is less than the allowed bet. However, sometimes the expected gain for a game in the series is significant, and you are allowed to make larger bets than the minimum.

Now what? You want to play because big bets on position expectation plays will bring good results overall, but you know that those bets which are greater than f* compromise the reasonable safety that goes with constant f* bets. To achieve the safety that f* units provide, you will have to reduce your bets to something less than optimal when expectation is positive, in order to offset the probable reduction in bankroll caused by bets made with poor expectation. The amount of bet reduction will depend on the degree of overbetting imposed by the nature of the particular compound game, and the player's readiness to abort a game that turns negative.

Multiple Payoff Games

Now for multiple payoff games, in which you may win or lose different multiples of your original bet. The expected rate of return (R) for an original bet of bankroll fraction (f) that will result in winning or losing various multiples (m) of f with various probabilities (p) is:

$$R = (1 + fm_1)^{p1} (1 + fm_2)^{p2} \ldots\ldots\ldots\ldots (1 + fm_n)^{pn} - 1$$

Some m's are negative, and the p's total 1.0

Finding f* is far from trivial here, involving calculus that is beyond my meager knowledge, but there is a shortcut that works well if the overall advantage is small--as is usually the case in the real world. Just calculate the variance of all the possible results for a one-unit bet, and divide the overall advantage by the variance. The result will be a bet fraction that is close to the theoretically correct f*. To get the variance, calculate the average of the squares of each possible result and subtract the square of the average result, with each result weighted by its probability.

Complex Games

A compound game with multiple payoffs per play is a complex game. The prime example is the casino game of 21, popularly called Blackjack. ("Blackjack" is really the home-type game, in which the participants take turns as dealer.) When the proportion of small cards dealt shows that the remaining cards favor the player, an optimal wager is in order. The methods for estimating the advantage are best described in *The Theory of Blackjack* by Peter A. Griffin.

Many players (and writers) use this advantage percentage as a bankroll proportion for an optimal bet. The proper approach is to divide the advantage by the variance to obtain an approximately correct f*. The variance for various 21 games in most situations is in the range of 1.3 to 1.4, so ignoring this factor can lead to some significant overbetting for the optimal bettor.

Since 21 is a compound game, it is wise to reduce bets even further to compensate for the excessive bets that are unavoidable for most styles of play. Two-thirds of the time the cards do not favor the player, so a "sit-down" player is overbetting his bankroll most of the time, even with minimum bets. The "table-hopper" (who leaves when the "count" goes negative) does better in avoiding these situations, while the "back-counter" (who lurks behind the table, waiting for a good count) never overbets. Each will have a different reduction factor for their optimal bets, which will also take into account the following inefficiencies:

1. The calculation of the advantage is necessarily approximate.

2. Bets must be rounded off to chip values.

3. Radical bet changes may not be tolerated by the "pit boss."